

# DEFECT OF COMPACTNESS IN SPACES OF BOUNDED VARIATION

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**ABSTRACT.** Defect of compactness for non-compact imbeddings of Banach spaces can be expressed in the form of a profile decomposition. Let  $X$  be a Banach space continuously imbedded into a Banach space  $Y$ , and let  $D$  be a group of linear isometric operators on  $X$ . A profile decomposition in  $X$ , relative to  $D$  and  $Y$ , for a bounded sequence  $(x_k)_{k \in \mathbb{N}} \subset X$  is a sequence  $(S_k)_{k \in \mathbb{N}}$ , such that  $(x_k - S_k)_{k \in \mathbb{N}}$  is a convergent sequence in  $Y$ , and, furthermore,  $S_k$  has the particular form  $S_k = \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)}$  with  $g_k^{(n)} \in D$  and  $w^{(n)} \in X$ . This paper extends the profile decomposition proved by Solimini [9] for Sobolev spaces  $\dot{H}^{1,p}(\mathbb{R}^N)$  with  $1 < p < N$  to the non-reflexive case  $p = 1$ . Since existence of “concentration profiles”  $w^{(n)}$  relies on weak-star compactness, and the space  $\dot{H}^{1,1}$  is not a conjugate of a Banach space, we prove a corresponding result for a larger space of functions of bounded variation. The result extends also to spaces of bounded variation on Lie groups.

## 1. INTRODUCTION

In presence of a compact imbedding of a reflexive Banach space  $X$  into another Banach space  $Y$ , Banach-Alaoglu theorem implies that any bounded sequence in  $X$  has a subsequence convergent in  $Y$ . If the imbedding  $X \hookrightarrow Y$  is continuous but not compact, it may be possible to characterize a suitable subsequence as convergent in  $X$  once one subtracts a suitable “defect of compactness”, which typically, for sequences of functions, isolates the singular behavior of the sequence. In broad sense this approach is known as *concentration compactness*, and in its more specific form, when the defect of compactness is expressed as a sum of elementary concentrations, is called profile decomposition. Profile decompositions were introduced by Michael Struwe in 1984 for particular class of sequences in Sobolev spaces.

**Definition 1.1.** Profile decomposition of a sequence  $(x_k)$  in a reflexive Banach space  $X$ , relative to a group  $D$  of isometries of  $X$ , is an asymptotic representation of  $x_k$  as a convergent sum  $S_k = \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)}$  with  $g_k^{(n)} \in D$ ,  $w^{(n)} \in X$ , such that  $g_k(x_k - S_k) \rightharpoonup 0$  for any sequence  $(g_k) \subset D$ . In the latter case one says that  $x_k - S_k$  converges to zero  $D$ -weakly.

We refer the reader for motivation of profile decomposition as an extension of the Banach-Alaoglu theorem, and a proof of both via non-standard analysis, to Tao [12]. For *general* bounded sequences in Sobolev spaces  $\dot{H}^{1,p}(\mathbb{R}^N)$ , the profile

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decomposition, relative to the group of translations and dilations, was proved in [9], and the  $D$ -weak convergence of the remainder was identified as convergence in the Lorentz spaces  $L^{p^*,q}$ ,  $q > p$ , where  $p^* = \frac{pN}{N-p}$ , and  $1 < p < N$  (which includes  $L^{p^*}$  but excludes  $L^{p^*,p}$ ). The result of [9] was later reproduced by Gérard [5] and Jaffard [6], who extended it to the case of fractional Sobolev spaces, but, on the other hand, gave a weaker form of remainder. For general Hilbert spaces, equipped with a non-compact group of isometries of particular type, existence of profile decomposition was proved in [8]. This, in turn, stimulated the search for new concentration mechanisms, i.e. different groups  $D$ , that yield profile decompositions in concrete functional spaces. In particular profile decompositions were proved with inhomogeneous dilations  $j^{-1/2}u(z^j)$ ,  $j \in \mathbb{N}$ , with  $z^j$  denoting an integer power of a complex number, for problems in the Sobolev space  $H_0^{1,2}(B)$  of the unit disk, related to the Trudinger-Moser functional; and with the action of the Galilean invariance, together with shifts and rescalings, involved in the loss of compactness in Strichartz imbeddings for the nonlinear Schrödinger equations. For a more comprehensive summary of known profile decompositions, including Besov and Triebel-Lizorkin spaces we refer the reader to a recent survey [14]. Profile decomposition in the general, uniformly convex and uniformly smooth, Banach space was proved recently in [11], which required to introduce a new mode of convergence of weak type ([10]). Not unlike [11], this paper studies profile decomposition by adapting the prior work on the topic to the mode of convergence of weak type which is pertinent in the new setting. A profile decomposition in the general non-reflexive Banach space remains an open problem, and one should note that no profile decomposition is possible when  $p = \infty$ , see e.g. a counterexample in [11].

The space  $L^1(\mathbb{R}^N)$  lacks weak (or weak-star) sequential compactness: indeed, consider a sequence of characteristic functions normalized in  $L^1$ ,  $(\frac{1}{|A_n|}\chi_{A_n})$ , where  $\mathbb{R}^N \supset A_1 \supset A_2 \supset \dots$  are closed nested sets with  $|\cap_{n \in \mathbb{N}} A_n| = 0$  which has no weakly convergent subsequence, while at the same time it converges weakly in the sense of measures to the Dirac delta-function. This suggests that when one studies a mapping on  $L^1(\mathbb{R}^N)$ , it may be beneficial to extend it to a larger domain, namely to the space of finite signed measures, which is a conjugate of the Banach space  $C_0(\mathbb{R}^N)$  and thus has the weak-\* compactness property. Similarly, it may be beneficial for a study of a mapping on the Sobolev space  $\dot{H}^{1,1}(\mathbb{R}^N)$  to extend its domain to the space of measurable functions whose weak derivative is a finite measure (rather than necessarily a  $L^1$ -function), in other words, to the space of functions of bounded variation. This space,  $\dot{BV}(\mathbb{R}^N)$  contains, of course, functions that are qualitatively different from those in  $H^{1,1}$ . For example, the characteristic function of a ball belongs to  $\dot{BV}(\mathbb{R}^N)$  and has a disconnected range  $\{0, 1\}$ , while every element in  $\dot{H}^{1,1}(\mathbb{R}^N)$  is represented by a function with a connected range.

The space of functions of bounded variations is of particular importance in geometric measure theory and in image analysis and has been a subject of intense scholarly interest. Our arguments generally follow the previous proofs of profile decompositions ([13, 9]) with adjustments to the peculiarities of the space. In Section 2 we summarize some known properties of the functions of bounded variation. In Section 3 we prove that imbedding into  $L^{N/(N-1)}$  is cocompact relative to the group of dyadic dilations and translations, which allows us in Section 4 to get a profile decomposition with the remainder vanishing in  $L^{N/(N-1)}$ . In Section 5 we give some applications for minimizations of functionals that complement the applications in

the paper of Bartsch and Willem [2], Theorem 5.2 in particular, as it requires a cocompactness argument. In Section 6 we generalize the profile decomposition of Section 4 to the case of subelliptic operators on nilpotent Lie groups. The main results of the paper are Theorem 3.2, Theorem 4.1, their Lie group generalizations Theorem 6.1 and Theorem 6.2, and results on existence of minimizers Theorem 5.1 and Theorem 5.2.

## 2. SPACE $\dot{BV}(\mathbb{R}^N)$

We summarize here some known properties of the space of functions with bounded variation. For a comprehensive exposition of the subject we refer the reader to the book [1]. We assume throughout the paper that  $N \geq 2$ .

**Definition 2.1.** The space of functions of bounded variation  $\dot{BV}(\mathbb{R}^N)$  is the space of all measurable functions  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  vanishing at infinity (i.e.  $\forall \epsilon > 0 \ |\{x \in \mathbb{R}^N : |u(x)| > \epsilon\}| < \infty$ ) such that

$$\|Du\| := \sup_{v \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^N) : \|v\|_\infty = 1} \int_{\mathbb{R}^N} u \operatorname{div} v < \infty. \quad (2.1)$$

The  $\dot{BV}(\mathbb{R}^N)$ -norm can be interpreted as the total variation  $\|Du\|$  of the measure associated with the derivative  $Du$  (in the sense of distributions on  $\mathbb{R}^N$ ). If  $u \in C_0^1(\mathbb{R}^N)$ , then the right hand side in (2.1) by integration by parts equals  $\int |\nabla u|$ . The value of the total variation of  $Du$  on a measurable set  $A \subset \mathbb{R}^N$  will be denoted as  $\|Du\|_A$ .

The space  $\dot{BV}(\mathbb{R}^N)$  is a conjugate space and therefore is complete. We will follow the convention that calls the weak-star convergence in the space of bounded variation *weak convergence*. It is well-known that  $BV(\mathbb{R}^N)$  is separable and therefore each bounded sequence in  $\dot{BV}(\mathbb{R}^N)$  has a weakly (i.e. weakly-star) convergent subsequence.

**Definition 2.2.** A sequence  $(u_k) \subset \dot{BV}(\mathbb{R}^N)$  is said to converge weakly to  $u$  if  $u_k \rightarrow u$  in  $L_{\text{loc}}^1(\mathbb{R}^N)$  and the weak derivatives  $\partial_i u_k$ ,  $i = 1, \dots, N$  converge to  $\partial_i u$  weakly as finite measures on  $\mathbb{R}^N$ .

We will need the following properties of  $\dot{BV}(\mathbb{R}^N)$ .

- (1) *Invariance.* The group of operators on  $\dot{BV}(\mathbb{R}^N)$ ,

$$D = \{g[j, y] : u \mapsto 2^{(N-1)j} u(2^j(\cdot - y))\}_{j \in \mathbb{Z}, y \in \mathbb{R}^N}, \quad (2.2)$$

consists of linear isometries of  $\dot{BV}(\mathbb{R}^N)$ , which are also linear isometries on  $L^{\frac{N}{N-1}}(\mathbb{R}^N)$ .

- (2) Density of  $C_0^\infty(\mathbb{R}^N)$  in *strict* topology (the closure of  $C_0^\infty(\mathbb{R}^N)$  in the norm topology is  $\dot{H}^{1,1}$ ). One says that  $u_k$  converges strictly to  $u$  if  $\|u_k - u\|_{1^*} \rightarrow 0$  and  $\|Du_k\| \rightarrow \|Du\|$ .
- (3) V.Maz'ya's inequality (often referred to as Sobolev, Aubin-Talenti or Gagliardo-Nirenberg inequality) [7]

$$NV_N^{1/N} \|u\|_{1^*} \leq \|Du\|, \quad (2.3)$$

where  $1^* = \frac{N}{N-1}$  and  $V_N$  is the volume of the unit ball in  $\mathbb{R}^N$ . A local version of this inequality is

$$\|u\|_{1^*, \Omega} \leq C(\|Du\|_\Omega + \|u\|_{1, \Omega}),$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain with sufficiently regular, say locally  $C^1$ -boundary.

(4) Hardy inequality :

$$\|Du\| \geq (N-1) \int_{\mathbb{R}^N} \frac{|u|}{|x|} dx.$$

(It follows from the Hardy inequality in  $\dot{H}^{1,1}$  and the density of  $C_0^\infty$  in  $\dot{BV}(\mathbb{R}^N)$  with respect to the strict convergence, if one first replaces  $1/|x|$  with its  $L^N$ -approximations from below.)

- (5) Local compactness: for any set  $\Omega \subset \mathbb{R}^N$  of finite Lebesgue measure,  $\dot{BV}(\mathbb{R}^N)$  is compactly imbedded into  $L^1(\Omega)$ .
- (6) Chain rule (a simplified version of a more elaborate statement due to Vol'pert, see [1, Remark 3.98]): let  $\varphi \in C^1(\mathbb{R})$ . Then for every  $u \in \dot{BV}(\mathbb{R}^N)$ ,

$$\|D\varphi(u)\| \leq \|\varphi'\|_\infty \|Du\|. \quad (2.4)$$

### 3. COCOMPACTNESS OF THE IMBEDDING $\dot{BV}(\mathbb{R}^N) \hookrightarrow L^{1^*}(\mathbb{R}^N)$

**Definition 3.1.** Let  $X$  be a Banach space and let  $D$  be a group of linear isometries of  $X$ . One says that a sequence  $(x_k) \subset X$  is  $D$ -vanishing (to be written  $x_k \xrightarrow{D} 0$ ) if for any sequence  $(g_k) \subset D$  one has  $g_k x_k \rightarrow 0$ . A continuous imbedding of  $X$  into a topological space  $Y$  is called cocompact with respect to  $D$  if  $x_k \xrightarrow{D} 0$  implies  $x_k \rightarrow 0$  in  $Y$ .

We extend this definition to  $\dot{BV}(\mathbb{R}^N)$  by understanding weak convergence in the sense of Definition 2.2.

The proof of the theorem below repeats much of the proof of Lemma 5.3 in [13], but uses different argument for the evaluation of sums of BV-seminorms over lattices.

**Theorem 3.2.** *The imbedding  $\dot{BV}(\mathbb{R}^N) \hookrightarrow L^{1^*}(\mathbb{R}^N)$  is cocompact relative to the group (2.2), i.e. if, for any sequence  $(j_k, y_k) \subset \mathbb{Z} \times \mathbb{R}^N$ ,  $g[j_k, y_k]u_k \rightarrow 0$  then  $u_k \rightarrow 0$  in  $L^{1^*}(\mathbb{R}^N)$ .*

*Proof.* Let  $(u_k) \subset \dot{BV}(\mathbb{R}^N)$  be such that for any  $(j_k, y_k) \subset \mathbb{Z} \times \mathbb{R}^N$ ,  $g[j_k, y_k]u_k \rightarrow 0$ .

1. Assume first that  $\sup_{k \in \mathbb{N}} \|u_k\|_\infty < \infty$  and  $\sup_{k \in \mathbb{N}} \|u_k\|_{1, \mathbb{R}^N} < \infty$ . Then, using the  $L^\infty$ -boundedness of  $(u_k)$  we have

$$\int_{(0,1)^N} |u_k|^{1^*} \leq C \left( \|Du_k\|_{(0,1)^N} + \left| \int_{(0,1)^N} u_k \right| \right) \left( \int_{(0,1)^N} |u_k| \right)^{1^*-1}.$$

Repeating this inequality for the domain of integration  $(0,1)^N + y$ ,  $y \in \mathbb{Z}^N$ , and adding the resulting inequalities over all  $y \in \mathbb{Z}^N$ , we have

$$\int_{\mathbb{R}^N} |u_k|^{1^*} \leq C(\|Du_k\|_{\mathbb{R}^N} + \|u_k\|_{1, \mathbb{R}^N}) \left( \sup_{y \in \mathbb{Z}^N} \int_{(0,1)^N} |u_k(\cdot - y)| \right)^{1^*-1}. \quad (3.1)$$

Here we use the fact that the sum  $\sum_{y \in \mathbb{Z}^N} \|Du_k\|_{(0,1)^N + y}$  can be split into  $3^N$  sums of variations over unions of cubes with disjoint closures, each of them, as follows from Definition 2.1 bound by  $\|Du_k\|_{\mathbb{R}^N}$ , which implies  $\sum_{y \in \mathbb{Z}^N} \|Du_k\|_{(0,1)^N + y} \leq 3^N \|Du_k\|_{\mathbb{R}^N}$ .

The last term in (3.1) converges to zero, since by the assumption  $g[j_k, y_k]u_k \rightharpoonup 0$  we have  $u_k(\cdot - y_k) \rightarrow 0$  in  $L^1((0, 1)^N)$  for any sequence  $(y_k) \subset \mathbb{R}^N$ .

2. We now abandon the restrictions imposed in the previous step on the sequence  $(u_k)$ . Let  $\chi \in C_0^\infty((\frac{1}{2^{N-1}}, 4^{N-1}))$  be such that  $\chi(t) = t$  whenever  $t \in [1, 2^{N-1}]$ . Let  $\chi_j(t) = 2^{(N-2)j} \chi(2^{-(N-1)j}|t|)$ ,  $j \in \mathbb{Z}$ ,  $t \in \mathbb{R}$ , and note that  $\|\chi'_j\|_\infty = \|\chi'\|_\infty$ . Consider now a general sequence  $(u_k) \subset \dot{B}V(\mathbb{R}^N)$  satisfying  $g[j_k, y_k]u_k \rightharpoonup 0$  for any  $(j_k, y_k) \subset \mathbb{Z} \times \mathbb{R}^N$ . By (2.3) we have

$$\int \chi_j(u_k)^{1^*} \leq C \|D\chi_j(u_k)\| \left( \int \chi_j(u_k)^{1^*} \right)^{1^*-1}.$$

Let us sum up the inequalities over  $j \in \mathbb{Z}$ . Note that by (2.4)  $\|D\chi_j(u_k)\| \leq \|\chi'\|_\infty \|Du_k\|_{A_{kj}}$  where  $A_{kj} = \{x \in \mathbb{R}^N : |u_k| \in (2^{(j-1)(N-1)}, 2^{(j+2)(N-1)})\}$ . Furthermore, one can break all the integers  $j$  into six disjoint sets  $J_1, \dots, J_6$ , such that, for any  $m \in \{1, 2, 3, 4, 5, 6\}$ , all functions  $\chi_j(u_k)$ ,  $j \in J_m$ , have pairwise disjoint supports. Consequently,  $\sum \|Du_k\|_{A_{kj}} \leq 6 \|Du_k\|$ . We have therefore

$$\int_{\mathbb{R}^N} |u_k|^{1^*} \leq C \|Du_k\| \sup_{j \in \mathbb{Z}} \left( \int \chi_j(u_k)^{1^*} \right)^{1^*-1}.$$

It suffices now to show that for any sequence  $(j_k) \subset \mathbb{Z}$ ,  $\chi_{j_k}(u_k) \rightarrow 0$  in  $L^{1^*}$ . Taking into account the invariance of the  $L^{1^*}$ -norm under operators  $g[j, y]$ , it suffices to show that  $\chi(2^{j_k(N-1)}|u_k(2^{j_k} \cdot)|) \rightarrow 0$  in  $L^{1^*}$ , but this is immediate from the assumption  $g[j_k, y_k]u_k \rightharpoonup 0$  and the argument of the step 1, once we take into account that for sequences uniformly bounded in  $L^\infty$ ,  $L^{1^*}$ -convergence follows from  $L^1$  convergence.  $\square$

**Corollary 3.3.** *The imbedding  $\dot{H}^{1,1}(\mathbb{R}^N) \hookrightarrow L^{1^*}(\mathbb{R}^N)$  is cocompact with respect to the group (2.2)*

#### 4. PROFILE DECOMPOSITION

**Theorem 4.1.** *Let  $(u_k) \subset \dot{B}V(\mathbb{R}^N)$  be a bounded sequence. For each  $n \in \mathbb{N}$  there exist  $w^{(n)} \in \dot{B}V(\mathbb{R}^N)$ , and sequences  $(j_k^{(n)}, y_k^{(n)}) \subset \mathbb{Z} \times \mathbb{R}^N$  with  $j_k^{(1)} = 0$ ,  $y_k^{(1)} = 0$ , satisfying*

$$|j_k^{(n)} - j_k^{(m)}| + |y_k^{(n)} - y_k^{(m)}| \rightarrow \infty \text{ whenever } m \neq n,$$

*such that for a renumbered subsequence,  $g[-j_k^{(n)}, -y_k^{(n)}]u_k \rightharpoonup w^{(n)}$ , as  $k \rightarrow \infty$ ,*

$$r_k \stackrel{\text{def}}{=} u_k - \sum_n g[j_k^{(n)}, y_k^{(n)}]w^{(n)} \rightarrow 0 \text{ in } L^{\frac{N}{N-1}}(\mathbb{R}^N), \quad (4.1)$$

*where the series  $\sum_n g[j_k^{(n)}, y_k^{(n)}]w^{(n)}$  converges in  $\dot{B}V(\mathbb{R}^N)$  uniformly in  $k$ , and*

$$\sum_{n \in \mathbb{N}} \|Dw^{(n)}\| + o(1) \leq \|Du_k\| \leq \sum_{n \in \mathbb{N}} \|Dw^{(n)}\| + \|Dr_k\| + o(1). \quad (4.2)$$

*Proof.* Without loss of generality we may assume that  $u_k \rightharpoonup 0$  (otherwise, one may pass to a weakly convergent subsequence and subtract the weak limit). Observe that if  $u_k \xrightarrow{D} 0$ , the theorem is proved with  $r_k = u_k$  and  $w^{(n)} = 0$ ,  $n \in \mathbb{N}$ . Otherwise consider the expressions of the form  $w^{(1)} = \text{w-lim } g[-j_k^{(1)}, -y_k^{(1)}]u_k$ . The sequence  $u_k$  is bounded,  $D$  is a set of isometries, so the sequence  $g[-j_k^{(1)}, -y_k^{(1)}]u_k$  has a weakly convergent subsequence. Since we assume that  $u_k$  is not  $D$ -vanishing, there exists

necessarily a sequence  $(j_k^{(1)}, y_k^{(1)})$  such that, evaluated on a suitable subsequence,  $w^{(1)} \neq 0$ . Let  $v_k^{(1)} = u_k - g[j_k^{(1)}, y_k^{(1)}]w^{(1)}$ , and observe that  $g[-j_k^{(1)}, -y_k^{(1)}]v_k^{(1)} = g[-j_k^{(1)}, -y_k^{(1)}]u_k - w^{(1)} \rightharpoonup 0$ . If  $v_k^{(1)} \xrightarrow{\mathcal{D}} 0$ , the assertion of the theorem is verified with  $r_k = v_k^{(1)}$ . If not - we repeat the argument above - there exist, necessarily, a sequence  $(j_k^{(1)}, y_k^{(1)})$  and a  $w^{(2)} \neq 0$  such that, on a renumbered subsequence,  $w^{(2)} = \text{w-lim } g[-j_k^{(2)}, -y_k^{(2)}]v_k^{(1)}$ . Let us set  $v_k^{(2)} = v_k^{(1)} - g[j_k^{(2)}, y_k^{(2)}]w^{(2)}$ . Then we will have

$$g[-j_k^{(2)}, -y_k^{(2)}]v_k^{(2)} = g[-j_k^{(2)}, -y_k^{(2)}]v_k^{(1)} - w^{(2)} \rightharpoonup 0.$$

If we assume that  $g[-j_k^{(1)}, -y_k^{(1)}]g[j_k^{(2)}, y_k^{(2)}]w^{(2)} \not\rightarrow 0$ , or, equivalently, that  $|j_k^{(1)} - j_k^{(2)}| + |y_k^{(1)} - y_k^{(2)}|$  has a bounded subsequence, then, passing to a renamed subsequence we will have  $g[-j_k^{(1)}, -y_k^{(1)}]g[j_k^{(2)}, y_k^{(2)}] \rightarrow g[j_0, y_0]$  in the sense of strong operator convergence, for some  $j_0 \in \mathbb{Z}$ ,  $y_0 \in \mathbb{R}^N$ . Then

$$\begin{aligned} w^{(2)} &= \text{w-lim } g[-j_k^{(2)}, -y_k^{(2)}]v_k^{(1)} = \text{w-lim } (g[-j_k^{(2)}, -y_k^{(2)}]g[j_k^{(1)}, y_k^{(1)}])g[-j_k^{(1)}, -y_k^{(1)}]v_k^{(1)} \\ &= \text{w-lim } (g[-j_0, -y_0]g[-j_k^{(1)}, -y_k^{(1)}]v_k^{(1)}) = 0, \end{aligned}$$

a contradiction that proves that  $g[-j_k^{(1)}, -y_k^{(1)}]g[j_k^{(2)}, y_k^{(2)}] \rightharpoonup 0$ , or, equivalently,  $|j_k^{(1)} - j_k^{(2)}| + |y_k^{(1)} - y_k^{(2)}| \rightarrow \infty$ . Then we also have  $g[-j_k^{(2)}, -y_k^{(2)}]g[j_k^{(1)}, y_k^{(1)}] \rightharpoonup 0$ .

Recursively we define:

$$v_k^{(n)} = v_k^{(n-1)} - g[j_k^{(n)}, y_k^{(n)}]w^{(n)} = u_k - g[j_k^{(1)}, y_k^{(1)}]w^{(1)} - \dots - g[j_k^{(n)}, y_k^{(n)}]w^{(n)},$$

where  $w^{(n)} = \text{w-lim } g[-j_k^{(n)}, -y_k^{(n)}]v_k^{(n-1)}$ , calculated on a successively renumbered subsequence. We subordinate the choice of  $(j_k^{(n)}, y_k^{(n)})$ , and thus the extraction of a subsequence for every given  $n$ , to the following requirements. For every  $n \in \mathbb{N}$  we set

$$W_n = \{w \in \dot{B}V(\mathbb{R}^N) \setminus \{0\} : \exists (j_k, y_k) \subset \mathbb{Z} \times \mathbb{R}^N, (k) \subset \mathbb{N}^{\mathbb{N}} : g[-j_m, -y_m]v_{k_m}^{(n)} \rightharpoonup w\},$$

and

$$t_n = \sup_{w \in W_n} \|Dw\|.$$

Note that  $t_n \leq \sup \|u_k\| < \infty$ . If for some  $n$ ,  $t_n = 0$ , the theorem is proved with  $r_k = v_k^{(n-1)}$ . Otherwise, we choose a  $w^{(n+1)} \in W_n$  such that  $\|Dw^{(n+1)}\| \geq \frac{1}{2}t_n$  and the sequence  $(j_k^{(n+1)}, y_k^{(n+1)})$  is chosen so that on a subsequence that we renumber,  $g[-j_k^{(n+1)}, -y_k^{(n+1)}]v_k^{(n)} \rightharpoonup w^{(n+1)}$ . An argument analogous to the one brought above for  $n = 1$  shows that  $g[-j_k^{(p)}, -y_k^{(p)}]g[j_k^{(q)}, y_k^{(q)}] \rightharpoonup 0$ , or, equivalently,

$$|j_k^{(p)} - j_k^{(q)}| + |y_k^{(p)} - y_k^{(q)}| \rightarrow \infty \quad (4.3)$$

whenever  $p \neq q$ ,  $p, q \leq n$ .

Let us show the lower bound inequality in (4.2). Let  $n \in \mathbb{N}$  and let  $(j_k^{(i)}, y_k^{(i)})_k$ ,  $w^{(i)}$  and  $(v_k^{(i)})_k$ ,  $i = 1, \dots, n$ , be defined as above. Let  $v^{(i)} \in C_0^\infty(\mathbb{R}^N)$ ,  $\|v^{(i)}\|_\infty \leq 1$   $i = 1, \dots, n$ , and set  $S_k^{(n)} = \sum_{i=1}^n g[j_k^{(i)}, y_k^{(i)}]w^{(i)}$ ,  $V_k^{(n)} = \sum_{i=1}^n 2^{(1-N)j_k^{(i)}} g[j_k^{(i)}, y_k^{(i)}]v^{(i)}$ . (To clarify the construction: the operator  $2^{(1-N)j/2}g[j, y]$  is the  $L^2(\mathbb{R}^N)$ -adjoint of  $g[-j, -y]$ .) Then, noting that  $\|V_k^{(n)}\|_\infty \leq 1$  and taking into account (4.3), we have

$$\|Du_k\| \geq \int v_k^{(n)} \text{div } V_k^{(n)} + \int S_k^{(n)} \text{div } V_k^{(n)}$$

$$= \sum_{i=1}^n \int g[-j_k^{(i)}, -y_k^{(i)}] v_k^{(n)} v^{(i)} + \sum_{i=1}^n w^{(i)} \operatorname{div} v^{(i)}.$$

Since the first term converges to zero by construction, while  $v^{(i)}$  is arbitrary, we have  $\|Du_k\| \geq \sum_{i=1}^n \|Dw^{(i)}\| + o_{k \rightarrow \infty}(1)$ . Since  $n$  is arbitrary, the lower bound in (4.2) follows.

Note now that  $\sum_{i=1}^\infty t_i \leq 2\|Du_k\| + o(1)$ . Furthermore,  $\|DS_k^{(n)}\| \leq \sum_{i=1}^n t_i + o(1)$ , and on a suitable subsequence we have  $\|DS_k^{(n)}\| \leq 2\sum_{i=1}^n t_i$ , and furthermore the inequality remains true even if one omits an arbitrary subset of terms in the sum  $S_k^{(n)}$ . Consequently, by an elementary diagonalization argument, on a suitable subsequence the series  $S_k^\infty$  converges in  $BV(\mathbb{R}^N)$  uniformly in  $k$ . This together with (4.3) implies that  $u_k - S_k^\infty \xrightarrow{D} 0$ , which by Theorem 3.2 implies (4.1). Finally, the second inequality in (4.2) follows from convergence of  $S_k^\infty$  and the triangle inequality for norms.  $\square$

## 5. SAMPLE MINIMIZATION PROBLEMS

Let  $a > 0$  be such that  $w := a\chi_B$ , where  $B$  is a unit ball, is a maximizer for the problem

$$c_0 = \sup_{u \in BV(\mathbb{R}^N): \|Du\|=1} \int_{\mathbb{R}^N} |u|^{1^*}.$$

By scaling invariance,  $w_R = R^{1-N}a\chi_{B_R}$  is then also a maximizer. In the following, generally non-compact, problem the existence of minimizers is proved by means of specific properties of the space  $BV(\mathbb{R}^N)$  rather than concentration argument.

**Theorem 5.1.** *Let function  $F \in C(\mathbb{R})$  be such that the following supremum is positive and is attained:*

$$0 < m = \sup_{s \in \mathbb{R}} F(s)/|s|^{1^*} = F(t)/|t|^{1^*} \text{ for some } t \in \mathbb{R}. \quad (5.1)$$

*Then the maximum in the relation*

$$c = \sup_{u \in BV(\mathbb{R}^N): \|Du\|=1} \int_{\mathbb{R}^N} F(u).$$

*is attained at the point  $w_R$  with  $R = (\frac{a}{t})^{\frac{1}{N-1}}$ .*

*Proof.* Since  $F(u) \leq m|u|^{1^*}$ , we have  $c \leq mc_0$ . On the other hand, comparing the supremum with the value of the functional at  $w_R$  we have  $c \geq \int F(w_R) = F(t)|B_R| = m|t|^{1^*}|B_R| = m \int |aR^{1-N}\chi_{B_R}|^{1^*} = mc_0$ . Therefore  $c = mc_0$  and is attained at  $w_R$ .  $\square$

**Theorem 5.2.** *Let  $0 < \lambda < N - 1$ . Then the minimum in*

$$\kappa = \inf_{u \in BV(\mathbb{R}^N): \int_{\mathbb{R}^N} |u|^{1^*} = 1} \|Du\| - \lambda \int \frac{|u|}{|x|} dx$$

*is attained.*

*Proof.* The proof of the argument is a standard use of profile decomposition and may be abbreviated. Let  $(u_k)$  be a minimizing sequence. Applying Theorem 4.1 and noting that there exists a subset of indices  $I \subset \mathbb{N}$  such that  $\int \frac{|u_k|}{|x|} dx \rightarrow \sum_{n \in I} \int \frac{|w^{(n)}|}{|x|} dx$

(provided that the functions  $w^{(n)}$  are redefined, as it is always possible, by application of constant operator  $g[j_n, y_n] \subset D$ ), we have, using the notation

$$J(u) = \|Du\| - \lambda \int \frac{|u|}{|x|} dx,$$

and recalling (4.2),

$$J(u_k) \geq \sum_{n \in I} J(w^{(n)}) + \sum_{n \notin I} \|Dw^{(n)}\| + o(1). \quad (5.2)$$

On the other hand, from Brezis-Lieb lemma follows

$$\int_{\mathbb{R}^N} |u_k|^{1^*} = \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^N} |w^{(n)}|^{1^*} + o(1). \quad (5.3)$$

Moreover, each  $w^{(n)}$  necessarily minimizes the respective functional ( $J$  if  $n \in I$ ,  $\|D \cdot\|$  if  $n \notin I$ ) over the functions  $u \in \dot{BV}(\mathbb{R}^N)$  satisfying  $\int_{\mathbb{R}^N} |u|^{1^*} = \int_{\mathbb{R}^N} |w^{(n)}|^{1^*}$ . In particular,  $w^{(n)}$  for  $n \notin I$  are multiples of the characteristic function of some ball, which are clearly not minimizers, up to normalization, for the functional  $J$ .

From the standard convexity argument, relations (5.2) and (5.3) imply that, necessarily,  $w^{(n)} = 0$  for all  $n \in \mathbb{N}$  except  $n = m$  with some  $m \in I$ . Thus,  $\int_{\mathbb{R}^N} |w^{(m)}|^{1^*} = \int_{\mathbb{R}^N} |u_k|^{1^*} = 1$  and  $J(w^{(m)}) \leq J(u_k) = \kappa + o(1)$ . This implies that  $w^{(m)}$  is a minimizer.  $\square$

### 6. 6.1 PROFILE DECOMPOSITION IN $\dot{BV}(G)$ FOR CARNOT GROUPS

The space of bounded variations on stratified nilpotent Lie groups, often called Carnot groups, has been studied in detail by Garofalo and Nhieu [4] and we first summarize relevant definitions and properties from that paper. The underlying Sobolev space is defined on a Carnot group  $G$  by a set of vector fields  $\{X_j\}_{j=1, \dots, n}$  which satisfy the Hörmander condition. More specifically, vectors  $\{X_j\}_{j=1, \dots, n}$  span the first stratum  $Y_1$  of the associated Lie algebra, their commutators spans the second stratum  $Y_2$ , and further successive commutations define further successive strata. Since the group is nilpotent, there is a minimal number  $m \geq 1$  such that  $Y_{m+1} = \{0\}$  and Hörmander condition is equivalent to  $Y_1 \dots Y_m$  spanning the whole Lie algebra. The left shift invariant Haar measure on such groups coincides with the Lebesgue measure. The case  $m = 1$  is the Euclidean case and the most commonly occurring example in literature is the Heisenberg group corresponding to  $N = 3$ ,  $n = 2$ , and  $m = 2$ . The subelliptic Sobolev space  $\dot{H}^{1,1}(G)$ ,  $n > 1$ , is the space of measurable functions such that

$$\|u\|_{1,1} = \int_G \sum_{j=1}^n |X_j u| d\lambda.$$

The related space of bounded variations  $\dot{BV}(G)$  is defined by the norm

$$\|Du\|_G = \sup_{v \in C_0^1(G; \mathbb{R}^N): \|v\|_\infty \leq 1} \int_G u \sum_{j=1}^n X_j^* v_j d\lambda. \quad (6.1)$$

There is a continuous imbedding  $\dot{BV}(G) \hookrightarrow L^{1^*}(G)$  where  $1^* = \frac{Q}{Q-1}$ , where  $Q = \sum_{i=1}^m i \dim Y_i$ .

A seminorm  $\|Du\|_\Omega$ , where  $\Omega \subset G$  is a Lebesgue-measurable set, is defined by the expression (6.1) with the integration over  $\Omega$  instead of  $G$ . When  $\Omega$  is a bounded



domain with Lipschitz boundary,  $\|Du\|_\Omega + \int_\Omega |u|$  is a norm defining the subelliptic Sobolev space  $BV(\Omega)$  which is compactly imbedded into  $L^1(\Omega)$ .

The simplified chain rule in  $\dot{B}V(G)$ , given a function  $f \in C^1(\mathbb{R})$ , is the inequality

$$\|Df(u)\|_\Omega \leq \|f'\|_\infty \|Du\|_\Omega.$$

We define the group  $D_G$  of isometries on  $\dot{B}V(G)$  as a product group of left shifts by  $G$ ,  $u \mapsto u \circ \eta$ ,  $\eta \in G$ , and of discrete dilations

$$h_j = u \mapsto 2^{(Q-1)j} u \circ \exp \circ \delta_{2^j} \circ \exp^{-1}, \quad j \in \mathbb{Z},$$

where the anisotropic dilations  $\delta_t$ ,  $t > 0$ , map the variables  $(y_1, \dots, y_m) \in Y_1 \times \dots \times Y_m$  of the stratified Lie algebra of  $G$  into  $ty_1, \dots, t^m y_m$ .

We have

**Theorem 6.1.** *Let  $G \simeq \mathbb{R}^N$ ,  $N \geq 2$ , be a stratified nilpotent Lie group. The imbedding  $\dot{B}V(G) \hookrightarrow L^{\frac{Q}{Q-1}}(G)$  is cocompact with respect to the operator group  $D_G$ . More specifically, if for any sequence  $(j_k, \eta_k) \subset \mathbb{Z} \times G$ ,  $(h_{j_k} u_k) \circ \eta_k \rightharpoonup 0$  then  $u_k \rightarrow 0$  in  $L^{\frac{Q}{Q-1}}(G)$ .*

The proof is a straightforward combination of the proof of Theorem 3.2 and the proof of an analogous statement for  $\dot{H}^{1,2}(G)$  ([13, Lemma 9.4], once notes the need in the covering lemma [13, Lemma A1].

**Theorem 6.2.** *Let  $G \simeq \mathbb{R}^N$ ,  $N \geq 2$ , be a stratified nilpotent Lie group. Let  $(u_k) \subset \dot{B}V(G)$  be a bounded sequence. For each  $n \in \mathbb{N}$  there exist  $w^{(n)} \in \dot{B}V(G)$ , and sequences  $(j_k^{(n)}, \eta_k^{(n)}) \subset \mathbb{Z} \times G$  with  $j_k^{(1)} = 0$ ,  $\eta_k^{(1)} = e$ , satisfying*

$$|j_k^{(n)} - j_k^{(m)}| + |\exp^{-1}(\eta_k^{(n)} \circ \eta_k^{(m)-1})| \rightarrow \infty \text{ whenever } m \neq n,$$

such that for a renumbered subsequence,  $(h_{-j_k} u_k) \circ \eta_k^{-1} \rightharpoonup w^{(n)}$ , as  $k \rightarrow \infty$ ,

$$r_k \stackrel{\text{def}}{=} u_k - \sum_n h_{j_k^{(n)}}(w^{(n)} \circ \eta_k^{(n)}) \rightarrow 0 \text{ in } L^{\frac{Q}{Q-1}}(G), \quad (6.2)$$

where the series  $\sum_n h_{j_k^{(n)}}(u_k \circ \eta_k^{(n)}) w^{(n)}$  converges in  $\dot{B}V(G)$  uniformly in  $k$ , and

$$\sum_{n \in \mathbb{N}} \|Dw^{(n)}\| + o(1) \leq \|Du_k\| \leq \sum_{n \in \mathbb{N}} \|Dw^{(n)}\| + \|Dr_k\| + o(1). \quad (6.3)$$

The proof is a straightforward modification of the proof of Theorem 4.1 on the lines of [13, Remark 9.3].

*Remark 6.3.* As we already mentioned, there is no profile decomposition in  $L^\infty$ , while the existing proof of profile decomposition in Banach spaces (in [11]) is based on a bound on the norms of profiles in the form

$$\sum_n \delta(\|w^{(n)}\|/\|u_k\|) \leq 1$$

that involves the modulus of convexity of the space, and it is not known in general which non-reflexive spaces admit profile decompositions. Is possible, however, in presence of an imbedding into a uniformly convex space (such as  $L^{1^*}$  in the present paper) to write a profile decomposition in terms of the target space, but this yields convergence of the series representing defect of compactness only in the weaker norm of the target space.

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